

SYMMETRIZATION WITH RESPECT TO THE ANISOTROPIC PERIMETER AND APPLICATIONS

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ABSTRACT. In this paper we introduce a new type of symmetrization, which preserves the anisotropic perimeter of the level sets of a suitable concave smooth function, in order to prove sharp comparison results for solutions of a class of homogeneous Dirichlet fully nonlinear elliptic problems of second order and for suitable anisotropic Hessian integrals.

Key words: Fully nonlinear elliptic equations, Pólya-Szegő inequality, Wulff shape, anisotropic symmetrization

2010 Mathematics Subject Classification: 35J25, 35A23

1. INTRODUCTION

Let Ω be a bounded, strictly convex domain of \mathbb{R}^2 . In a nowadays classical paper by Talenti [Ta], it was proved that if u is a C^2 concave solution of the Monge-Ampère equation

$$\begin{cases} \det \nabla^2 u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

with f positive and sufficiently smooth function, then

$$\tilde{u}(x) \leq v(x), \quad x \in D, \quad (1.2)$$

where v is the positive concave solution of the problem

$$\begin{cases} \det \nabla^2 v = f^\#(x) & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \quad (1.3)$$

Here D is the disk centered at the origin with the same perimeter of Ω , $f^\#$ is the spherically decreasing rearrangement of f , and \tilde{u} is the spherically decreasing function in D whose level sets have the same perimeter of the level sets of u . Hence, among all the problems of the type (1.1) with prescribed perimeter of Ω and fixed rearrangement of f , problem (1.3) gives the “maximal” solution. After this result, generalizations in several directions have been studied (see for example [Br, BNT, BT, BT2, DG3, DG4, Ga, Tr, Ts]). Moreover, it was proved in [Tr], [Ts] that a Pólya-Szegő type inequality for the Monge-Ampère operator holds, namely

$$\int_{\Omega} u \det D^2 u \, dx \geq \int_D \tilde{u} \det D^2 \tilde{u} \, dx. \quad (1.4)$$

Hence, the symmetrization with respect to the perimeter decreases the Hessian integral.

In this paper we take into account a class of fully nonlinear elliptic anisotropic operators of the second order, which contains the classical Monge-Ampère operator. More precisely, we consider a sufficiently smooth norm H of \mathbb{R}^2 , and denote with H° its polar function (see Section 2 for the precise assumptions). If u a smooth concave function, we take into account the following anisotropic Monge-Ampère operator

$$\det_H[u] := \det \nabla_\xi^2 F \det \nabla^2 u, \quad (1.5)$$

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where $F(\xi) = \frac{1}{2}H(\xi)^2$, $\xi \in \mathbb{R}^2$, and $\nabla_\xi^2 F(\xi)$ is the 2×2 matrix of the second derivatives of F . We observe that when H is the Euclidean norm of \mathbb{R}^2 , then the matrix $\nabla_\xi^2 F$ reduces to the identity, and (1.5) reduces with the classical Monge-Ampère operator.

Several kind of problems related to anisotropic operators have been largely studied in last years. We refer the reader, for example, to [ALFT, ATW, An, BP, BFK, BF, CS, CFV, DG2, DG5, ET, FV, Ja, WX].

The aim of this paper is to prove a suitable generalization of the inequalities (1.2) and (1.4) for the operator (1.5) using a new type of symmetrization. More precisely, given a concave smooth function u , the rearranged function we introduce preserves the anisotropic perimeter of its level sets, that is a suitable measure of the length of $\{u = t\}$ which takes into account the anisotropy H . For this purpose a fundamental tool is a well-known anisotropic isoperimetric inequality (see for instance [Bu, ALFT, FM, DP, DG]). Moreover, a key role is played by a relation between the operator (1.5) and the anisotropic curvature of the level sets of u (see Sections 2 and 3).

The structure of the paper is the following. In Section 2, we state the main hypotheses on the norm H , and we recall the definition of anisotropic perimeter and curvature, proving a version of the Gauss-Bonnet Theorem in this setting. In Section 3, we introduce the anisotropic Monge-Ampère operator, proving its connection with the anisotropic curvature. Moreover, we compute the operator (1.5) for functions which are symmetric with respect to H^o .

The Section 4 is devoted to define the symmetrization with respect to the anisotropic perimeter and to prove the Pólya-Szegő inequality. Finally, in Section 5 we prove the quoted comparison result in the spirit of Talenti's inequality.

2. NOTATION AND PRELIMINARIES

Throughout the paper we will denote by $H : \mathbb{R}^2 \rightarrow [0, +\infty[$, a convex function such that

$$\alpha|\xi| \leq H(\xi), \quad \forall \xi \in \mathbb{R}^2, \quad (2.1)$$

and

$$H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^2, \quad \forall t \in \mathbb{R}. \quad (2.2)$$

Under these hypotheses it is easy to see that there exists $\beta \geq \alpha$ such that

$$H(\xi) \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^2. \quad (2.3)$$

Moreover, we assume that H^2 is strongly convex, that is $H \in C^2(\mathbb{R}^2 \setminus \{0\})$ and the Hessian matrix $\nabla_\xi^2 H^2$ is positive definite in $\mathbb{R}^2 \setminus \{0\}$.

The polar function $H^o : \mathbb{R}^2 \rightarrow [0, +\infty[$ of H is defined as

$$H^o(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{H(\xi)} \quad (2.4)$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of \mathbb{R}^2 . It is easy to verify that also H^o is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$H(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{H^o(\xi)}.$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^n : H^o(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin. We put $\kappa = |\mathcal{W}|$, where $|\mathcal{W}|$ denotes the Lebesgue measure of \mathcal{W} . More generally, we denote with $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at x_0 with measure κr^2 , and $\mathcal{W}_r(0) = \mathcal{W}_r$.

The strong convexity of H^2 implies that $\{\xi \in \mathbb{R}^2 : H(\xi) < 1\}$ is strictly convex. This ensures that $H^o \in C^1(\mathbb{R}^2 \setminus \{0\})$. Actually, the strict convexity of the level sets of H is equivalent to the continuous differentiability of H^o in $\mathbb{R}^2 \setminus \{0\}$ (see [Sc] for the details).

The following properties of H and H^o hold true (see for example [BP]):

$$H(\xi) = H_\xi(\xi) \cdot \xi, \quad H^o(\xi) = H_\xi^o(\xi) \cdot \xi, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\} \quad (2.5)$$

$$H(H_\xi^o(\xi)) = H^o(H_\xi(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \quad (2.6)$$

$$H^o(\xi)H_\xi(H_\xi^o(\xi)) = H(\xi)H_\xi^o(H_\xi(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}. \quad (2.7)$$

2.1. Anisotropic perimeter. Let K be an open bounded set of \mathbb{R}^n with Lipschitz boundary. The perimeter of K is defined as the quantity

$$P_H(K) = \int_{\partial K} H(\nu_K) d\mathcal{H}^1,$$

where ν_K is the Euclidean outer normal to ∂K . For example, if $K = \mathcal{W}_R$, then

$$\begin{aligned} P_H(\mathcal{W}_R) &= \int_{\partial \mathcal{W}_R} \frac{1}{|\nabla H^o(x)|} d\mathcal{H}^1 = \frac{1}{R} \int_{\partial \mathcal{W}_R} \frac{x \cdot \nabla H^o(x)}{|\nabla H^o(x)|} d\mathcal{H}^1 = \\ &= \frac{1}{R} \int_{\partial \mathcal{W}_R} x \cdot \nu d\mathcal{H}^1 = \frac{2}{R} \int_{\mathcal{W}_R} dx = 2\kappa R, \end{aligned}$$

where in the above computations we used (2.5) and the divergence theorem.

The anisotropic perimeter of a set K is finite if and only if the usual Euclidean perimeter $P(K)$ is finite. Indeed, by properties (2.2) and (2.1) we have that

$$\frac{1}{\beta}|\xi| \leq H^o(\xi) \leq \frac{1}{\alpha}|\xi|,$$

and then

$$\alpha P(K) \leq P_H(K) \leq \beta P(K).$$

An isoperimetric inequality for the anisotropic perimeter holds, namely

$$P_H(K)^2 \geq 4\kappa|K| \quad (2.8)$$

(see for example [Bu, DP, FM, ALFT]). We stress that in [DG] an isoperimetric inequality for the anisotropic relative perimeter in the plane is studied.

Moreover, if K is a convex body of \mathbb{R}^2 , and $\delta > 0$, the following Steiner formulas hold (see [An, Sc]):

$$|K + \delta \mathcal{W}| = |K| + P_H(K)\delta + \kappa\delta^2$$

and

$$P_H(K + \delta \mathcal{W}) = P_H(K) + 2\kappa\delta. \quad (2.9)$$

2.2. Anisotropic curvature. For the sake of simplicity, will assume the following conventional notation: given a smooth function u , then $\partial_{x_i} u = u_i$, for $i = 1, 2$, and $\partial_{x_i x_j} u = u_{ij}$, for $i, j = 1, 2$.

We recall the definition and some properties of anisotropic curvature for a smooth set. For further details we refer the reader, for example, to [ATW] and [BP].

Definition 2.1. Let $K \subset \mathbb{R}^2$ be a bounded open set with smooth boundary, $\nu_K(x)$ the unit outer normal at $x \in \partial K$, in the usual Euclidean sense. Let u be a $C^2(K)$ function such that $u > 0$ on ∂K , $\partial K = \{u = 0\}$ and $\nabla u \neq (0, 0)$ on ∂K . Hence, $\nu_K = -\frac{\nabla u}{|\nabla u|}$ on ∂K . The anisotropic outer normal n_K is defined as

$$n_K(x) = \nabla_\xi H(\nu_K(x)) = \nabla H_\xi \left(-\frac{\nabla u}{|\nabla u|} \right), \quad x \in \partial K.$$

By the properties of H and H^o ,

$$H^o(n_K) = 1.$$

The anisotropic curvature k_H of ∂K is

$$k_H(x) = \operatorname{div} n_K(x) = \operatorname{div} \left[\nabla_\xi H \left(-\frac{\nabla u}{|\nabla u|} \right) \right], \quad x \in \partial K.$$

Being $\nabla_\xi H$ a 0-homogeneous function, it follows that

$$k_H(x) = -\operatorname{div} [\nabla_\xi H(\nabla u)] = -\sum_{i,j} H_{\xi_i \xi_j}(\nabla u) u_{ij} \quad \text{on } \partial K. \quad (2.10)$$

Remark 2.1. If $H(x) = (x_1^2 + x_2^2)^{1/2}$ is the Euclidean norm of \mathbb{R}^2 , then the above definition coincides with the classical definition of curvature in the plane.

Remark 2.2. We stress that if $K = \mathcal{W}_\lambda(x_0) = \{x \in \mathbb{R}^2 : H^o(x - x_0) < \lambda\}$, that is homothetic to the Wulff shape \mathcal{W} and centered at $x_0 \in \mathbb{R}^2$, the anisotropic outer normal at $x \in \partial K$ has the direction of $x - x_0$. Indeed by the properties of H it follows that

$$n_K(x) = \nabla_\xi H(\nabla_\xi H^o(x - x_0)) = \frac{1}{\lambda}(x - x_0), \quad x \in \partial K$$

(see Figure 1 for an example). Moreover, computing the anisotropic curvature at $x \in \partial K$ we have that

$$k_H(x) = \frac{1}{\lambda}.$$

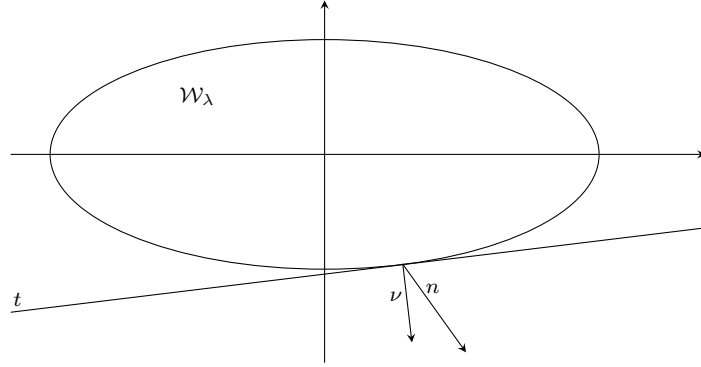


FIGURE 1. Here $H(x_1, x_2) = (x_1^2/a^2 + x_2^2/b^2)^{1/2}$ and $H^o(x_1, x_2) = (a^2x_1^2 + b^2x_2^2)^{1/2}$. When $a \neq b$, the usual and the anisotropic outer normal are, in general, different.

The anisotropic curvature and the anisotropic perimeter are related as follows. By computing the first variation of the perimeter (see [BP, Theorem 5.1], or [An, Section 2.6, formula (2.24)]) we have that if K has smooth boundary

$$\lim_{\delta \rightarrow 0^+} \frac{P_H(K + \delta \mathcal{W}) - P_H(K)}{\delta} = \int_{\partial \Omega} k_H(x) H(\nu) d\mathcal{H}^1.$$

Combining this identity and formula (2.9), we obtain the following anisotropic version of the Gauss-Bonnet theorem for boundaries of smooth convex sets.

Proposition 2.1. *If K is a convex, bounded open set such that $\partial K \in C^2$, then*

$$\int_{\partial K} k_H(x) H(\nu) d\mathcal{H}^1 = 2\kappa. \quad (2.11)$$

3. ANISOTROPIC MONGE-AMPÈRE OPERATOR

Given any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we denote by $F(\xi)$ the function

$$F(\xi) = \frac{1}{2} H(\xi)^2,$$

and by $\nabla_\xi^2 F(\xi)$ the 2×2 matrix of the second derivatives of F . Hence its components are

$$F_{\xi_i \xi_j} = H_{\xi_i} H_{\xi_j} + H H_{\xi_i \xi_j}, \quad \text{for } i, j = 1, 2.$$

Let u be a smooth function, and consider the fully nonlinear operator $u \mapsto A[u] = A(\nabla u, \nabla^2 u)$ defined as

$$A[u] = \nabla_x [F_\xi(\nabla u)] = \nabla_\xi^2 F(\nabla u) \nabla^2 u.$$

We will take into account equations whose principal part is the following:

$$\det_H[u] := \det A[u] = \det \nabla_\xi^2 F \det \nabla^2 u.$$

Remark 3.1. We point out that if $H(x) = (x_1^2 + x_2^2)^{1/2}$ is the Euclidean norm of \mathbb{R}^2 , then the matrix $\nabla_\xi^2 F$ reduces to the identity, and the operator $\det A[u]$ coincides with the classical Monge-Ampère operator.

We will consider a convex, bounded, open set $\Omega \in \mathbb{R}^2$ with C^2 boundary, and functions belonging to the class

$$\Phi_0(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \mid u \in W^{2,2}(\Omega) \cap C^1(\bar{\Omega}), u \equiv 0 \text{ on } \partial\Omega, u \text{ concave in } \Omega\}.$$

Being F strongly convex, the functions $u \in \Phi_0(\Omega)$ are admissible in order to have that \det_H is elliptic. Obviously, $u \in \Phi_0(\Omega)$ is either positive in Ω , or $v \equiv 0$ in $\bar{\Omega}$.

Let us denote by

$$(S^{ij}(B))_{ij} = \text{Cof } B = \begin{pmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} \end{pmatrix}$$

the cofactor of the matrix $B = (b_{ij})$. Observe that

$$\sum_{i,j} S^{ij}(A[u]) F_{\xi_i} u_j = \nabla_\xi F \text{Cof } A[u] \cdot \nabla u.$$

In [CS] the following integration by parts formula is proved.

Lemma 3.1. *Let $u \in W^{2,2}(\Omega) \cap C^1(\bar{\Omega})$, with Ω bounded open set such that $\partial\Omega \in C^1$, and $u = 0$ on $\partial\Omega$. Then*

$$\int_\Omega u \det_H[u] dx = -\frac{1}{2} \sum_{i,j} \int_\Omega S^{ij}(A[u]) F_{\xi_i}(\nabla u) u_j dx. \quad (3.1)$$

For a function $u \in \Phi_0(\Omega)$, we denote the set E_u as the following:

$$E_u = \{x \in \Omega : 0 \leq u(x) < \max_{\bar{\Omega}} u\}. \quad (3.2)$$

Being $u \in C^1(\bar{\Omega})$, E_u is an open set.

Theorem 3.1. *Let $u \in \Phi_0(\Omega) \cap C^2(E_u)$, $u \not\equiv 0$, and take $x \in \bar{\Omega}$ such that $u(x) \in [0, \max_{\bar{\Omega}} u[$.*

Then

$$k_H(x) = -H(\nabla u(x))^{-3} \sum_{i,j} S^{ij}(A[u(x)]) F_{\xi_i}(\nabla u(x)) u_j(x), \quad (3.3)$$

where $k_H(x)$ is the anisotropic curvature of the level set $\{y \in \Omega : u(y) = u(x)\}$ at the point x .

Proof. First of all, we observe that $|\nabla u(x)| \neq 0$ by the regularity and the concavity of $u \not\equiv 0$. Denoting $H_{\xi_i} = H_i$, $H_{\xi_i \xi_j} = H_{ij}$, and using the Einstein summation convention, we have

$$(S^{ij}(A[u]))_{ij} = \begin{pmatrix} H_2 H_m u_{2m} & -H_2 H_m u_{1m} \\ -H_1 H_m u_{2m} & H_1 H_m u_{1m} \end{pmatrix} + H \begin{pmatrix} H_{2m} u_{2m} & -H_{2m} u_{1m} \\ -H_{1m} u_{2m} & H_{1m} u_{1m} \end{pmatrix}.$$

Then, recalling that $k_H = -H_{ij}u_{ij}$, we have

$$\begin{aligned}
& S^{ij}(A[u])F_{\xi_i}(\nabla u)u_j \\
&= H^2(\nabla u) [H_{2m}u_{2m}H_1u_1 + H_{1m}u_{1m}H_2u_2 - H_{2m}u_{1m}H_1u_2 - H_{1m}u_{2m}H_2u_1] \\
&= H^2(\nabla u) [-k_H(H_1u_1 + H_2u_2) + \\
&\quad -H_1u_{11}(H_{11}u_1 + H_{12}u_2) - H_1u_{12}(H_{21}u_1 + H_{22}u_2) + \\
&\quad -H_2u_{12}(H_{11}u_1 + H_{12}u_2) - H_2u_{22}(H_{21}u_1 + H_{22}u_2)] \\
&= H^3(\nabla u) k_H(x),
\end{aligned}$$

where last equality follows from the 1 and 0 homogeneity of H and H_ξ , respectively, being $\nabla_\xi H_i(\nabla u) \cdot \nabla u = 0$. \square

Remark 3.2. We stress that if $H(x) = (x_1^2 + x_2^2)^{1/2}$ is the Euclidean norm of \mathbb{R}^2 , then the identity (3.3) reduces to the well-known formula for the Euclidean curvature of the level sets of u :

$$k(x) = -|\nabla u|^{-3} S^{ij}(\nabla^2 u) u_i u_j,$$

with

$$(S^{ij}(\nabla^2 u))_{ij} = \begin{pmatrix} u_{22} & -u_{12} \\ -u_{12} & u_{11} \end{pmatrix}.$$

The following Reilly-type inequality for the anisotropic determinant holds.

Proposition 3.1. *Let $u \in \Phi_0(\Omega) \cap C^2(E_u)$, where E_u is defined in (3.2) and Ω is a bounded convex open set such that $\partial\Omega \in C^2$. Then*

$$\int_{u>t} \det_H[u] dx = \frac{1}{2} \int_{u=t} k_H(x) \frac{H(\nabla u)^3}{|\nabla u|} d\mathcal{H}^1, \quad t \in [0, \max_\Omega u[. \quad (3.4)$$

Proof. Let $t \in [0, \max_\Omega u[$, and $h > 0$ sufficiently small, and apply Lemma 3.1 to the functions $u - t$ and $u - t - h$ in the sets $\{u > t\}$ and $\{u > t + h\}$ respectively. By subtracting, and recalling (3.3) we have

$$\begin{aligned}
\int_{t < u \leq t+h} (u - t) \det_H[u] dx - h \int_{u > t+h} \det_H[u] dx &= \frac{1}{2} \int_{t < u \leq t+h} k_H(x) H(\nabla u)^3 dx = \\
&= \frac{1}{2} \int_t^{t+h} d\tau \int_{u=\tau} k_H(x) \frac{H(\nabla u)^3}{|\nabla u|} d\mathcal{H}^1,
\end{aligned}$$

where last equality follows from the coarea formula. Hence, dividing for h and passing to the limit, we easily get (3.4). \square

3.1. The anisotropic Monge-Ampère operator for radial functions. Let $v(x) = w(H^o(x)) = w(r)$, $x \in \mathcal{W}_R$, $r = H^o(x)$. We compute the operator \det_H on v . We have:

$$\nabla v(x) = w'(r) H_\xi^o(x).$$

Then, using the homogeneity of H and properties (2.6) and (2.7), it follows that

$$F_\xi(\nabla v(x)) = w'(r) H(H_\xi^o(x)) H_\xi(H_\xi^o(x)) = w'(r) \frac{x}{H^o(x)}.$$

Then

$$A[v] = \begin{pmatrix} w'' \frac{x_1 H_{\xi_1}^o}{r} + \frac{w'}{r^2} \left(r - x_1 H_{\xi_1}^o \right) & \frac{x_2 H_{\xi_1}^o}{r} \left(w'' - \frac{w'}{r} \right) \\ \frac{x_1 H_{\xi_2}^o}{r} \left(w'' - \frac{w'}{r} \right) & w'' \frac{x_2 H_{\xi_2}^o}{r} + \frac{w'}{r^2} \left(r - x_2 H_{\xi_2}^o \right) \end{pmatrix}$$

Then, computing the determinant of $A[v]$ and using (2.5), a straightforward computation leads to

$$\det A[v] = \frac{w'w''}{r} = \frac{[(w')^2]'}{2r}. \quad (3.5)$$

Hence the function

$$v(x) = \sqrt{2} \int_{H^o(x)}^R \left(\int_0^s r g(r) dr \right)^{\frac{1}{2}} ds$$

is such that $v \in \Phi_0(\mathcal{W}_R) \cap C^2(\mathcal{W}_R \setminus \{0\})$ and v is the unique anisotropic radially symmetric function such that

$$\begin{cases} \det_H[v] = g(H^o(x)) & \text{a.e. in } \mathcal{W}_R \setminus \{0\} \\ v = 0 & \text{on } \partial\mathcal{W}_R. \end{cases}$$

4. SYMMETRIZATION WITH RESPECT TO THE ANISOTROPIC PERIMETER

Now we recall some basic definition on rearrangements and convex symmetrization. Moreover for a given function u , we introduce a new kind of symmetrization which preserves the anisotropic perimeter of the level sets of u .

Let Ω be a bounded open set, and $u: \Omega \rightarrow \mathbb{R}$ a measurable function. We will adopt the following notation:

$$\Omega_t = \{x \in \Omega: |u(x)| > t\} \text{ and } \Sigma_t = \partial\Omega_t = \{x \in \Omega: |u(x)| = t\}.$$

Moreover, $\mu(t) = |\Omega_t|$, $t \geq 0$ is the distribution function of u .

The decreasing rearrangement of u is the map $u^*: [0, \infty[\rightarrow \mathbb{R}$ defined by

$$u^*(s) := \sup\{t \in \mathbb{R} : \mu(t) > s\}.$$

The function u^* is the generalized inverse of μ .

Following [ALFT], the convex symmetrization of u is the function $u^\diamond(x)$, $x \in \Omega^\diamond$ defined by:

$$u^\diamond(x) = u^*(\kappa H^o(x)^2),$$

where Ω^\diamond is a set homothetic to the Wulff shape having the same measure of Ω .

Now suppose that Ω is a convex set of \mathbb{R}^2 and let $u \geq 0$ be a measurable function with convex level sets. For $t \in [0, \max_{\bar{\Omega}} u]$ the anisotropic perimeter of the level set Ω_t is denoted with

$$\lambda_H(t) = P_H(\Omega_t). \quad (4.1)$$

It is well-known that u , u^\diamond and u^* are equimeasurable.

Definition 4.1. The rearrangement of u with respect to the anisotropic perimeter is the function $s \in [0, P_H(\Omega)] \mapsto u^\star(s) \in [0, \max_{\bar{\Omega}} u]$ defined as

$$u^\star(s) = \sup\{t \geq 0: \lambda_H(t) \geq s\}.$$

Moreover, we define the anisotropic radial symmetrand of u with respect to the anisotropic perimeter the function

$$u^\star(x) = u^\star(2\kappa H^o(x)), \quad x \in \Omega^\star,$$

where Ω^\star is the set homothetic to the Wulff shape \mathcal{W} such that $P_H(\Omega^\star) = P_H(\Omega)$. More precisely, $\Omega^\star = \mathcal{W}_R$, with $R = \frac{P_H(\Omega)}{2\kappa}$.

From now on, we will suppose that $u \in \Phi_0(\Omega) \cap C^2(E_u)$, where E_u is defined in (3.2).

The functions u^\star and u^\diamond have the following properties:

- (i) u^\star is a concave and decreasing function in $[0, P_H(\Omega)]$;
- (ii) $u^\diamond(x)$ is symmetric and decreasing with respect to H^o ;
- (iii) The sets $\{u^\star > t\}$ are homothetic to the Wulff shape such that $P_H(\{u^\star > t\}) = P_H(\{u > t\})$.
- (iv) $u^\star(\lambda_H(t)) = t$.

If $u \in \Phi_0(\Omega)$, the coarea formula gives that

$$\mu'(t) = - \int_{\Sigma_t} \frac{1}{|\nabla u|} d\mathcal{H}^1, \quad t \in [0, \max_{\bar{\Omega}} u[. \quad (4.2)$$

Moreover, we have the following result.

Proposition 4.1. *If $u \in \Phi_0(\Omega) \cap C^2(E_u)$, the function $\lambda_H(t)$ defined in (4.1) is strictly decreasing in $[0, \max_{\bar{\Omega}} u]$, it is differentiable in $[0, \max_{\bar{\Omega}} u[$ and its derivative is*

$$\lambda'_H(t) = - \int_{\Sigma_t} \frac{k_H(x)}{|\nabla u|} d\mathcal{H}^1. \quad (4.3)$$

Proof. Using the homogeneity of H and the divergence theorem we get

$$\begin{aligned} \lambda_H(t) &= \int_{\Sigma_t} H(\nu_{\Omega_t}) d\mathcal{H}^1 = - \int_{\Sigma_t} \nabla_{\xi} H(\nabla u) \cdot \nu_{\Omega_t} d\mathcal{H}^1 = \\ &= - \int_{\Omega_t} \operatorname{div} \nabla_{\xi} H(\nabla u) dx = \int_{\Omega_t} k_H(x) dx. \end{aligned}$$

Hence, being $|\nabla u| \neq 0$ on Σ_t , for any $t \in [0, \max_{\bar{\Omega}} u[$, by the coarea formula we obtain, for $t \in [0, \max_{\bar{\Omega}} u[$, that

$$\frac{1}{h} [\lambda_H(t) - \lambda_H(t+h)] = \frac{1}{h} \int_{\{t < u \leq t+h\}} k_H(x) dx = \frac{1}{h} \int_t^{t+h} \int_{\Sigma_t} \frac{k_H(x)}{|\nabla u|} d\mathcal{H}^1.$$

Passing to the limit, we get (4.3). \square

As a consequence of Proposition 4.1, we have that

Proposition 4.2. *The function u^{\star} belongs to $C^{0,1}([0, P_H(\Omega)])$, and there exists a positive constant $C > 0$ such that*

$$0 \leq -(u^{\star})'(s) \leq C, \quad \text{for any } s \in]0, P_H(\Omega)].$$

Proof. Let $t \in [0, \max_{\bar{\Omega}} u[$. By Proposition 4.1, (2.3) and formula (2.11), we have

$$\begin{aligned} -\lambda'_H(t) &= \int_{\Sigma_t} \frac{k_H(x)}{|\nabla u|} d\mathcal{H}^1 \geq \frac{1}{\max_{\bar{\Omega}} |\nabla u|} \int_{\Sigma_t} k_H(x) d\mathcal{H}^1 \\ &\geq \frac{1}{\beta \max_{\bar{\Omega}} |\nabla u|} \int_{\Sigma_t} k_H(x) H(\nu) d\mathcal{H}^1 \\ &= \frac{2\kappa}{\beta \max_{\bar{\Omega}} |\nabla u|}. \end{aligned} \quad (4.4)$$

Being $u^{\star}(\lambda_H(t)) = t$ then

$$\lambda'_H(t) = \frac{1}{\frac{d}{dt} u^{\star}(\lambda_H(t))}.$$

Substituting in (5.1) we get the thesis. \square

The main difference between the symmetrand of u with respect to the anisotropic perimeter u^{\star} and the convex symmetrand of u , $u^{\diamond}(x)$, is that, in general, the first one increases the Lebesgue norms of u . Indeed, we have the following.

Proposition 4.3. *Let be $u \in \Phi_0(\Omega) \cap C^2(E_u)$ and u^{\star} as in Definition 4.1. Then*

$$\|u\|_{L^p(\Omega)} \leq \|u^{\star}\|_{L^p(\Omega^{\star})}, \quad 1 \leq p < +\infty,$$

and

$$\|u\|_{L^{\infty}(\Omega)} = \|u^{\star}\|_{L^{\infty}(\Omega^{\star})}.$$

Proof. It is enough to observe that, by the anisotropic isoperimetric inequality (2.8) we have

$$|\{u > t\}| = \mu(t) \leq \frac{\lambda_H^2(t)}{4\kappa} = \frac{P_H^2(\{u^\star > t\})}{4\kappa} = |\{u^\star > t\}|.$$

□

In order to prove a Pólya-type inequality for the symmetrization with respect to the anisotropic perimeter we need the following definition.

Definition 4.2 (Anisotropic Hessian integral). Let $u \in \Phi_0(\Omega) \cap C^2(E_u)$. Then the anisotropic Hessian integral is

$$I_H[u, \Omega] = \int_{\Omega} u \det_H[u] dx$$

Remark 4.1. By Theorem 3.1 and the identity (3.1), for $u \in \Phi_0(\Omega) \cap C^2(E_u)$ the anisotropic Hessian integral can be written also as follows

$$I_H[u, \Omega] = \int_{\Omega} u \det_H[u] dx = \frac{1}{2} \int_{\Omega} k_H(x) H^3(\nabla u) dx. \quad (4.5)$$

When we consider anisotropic radially symmetric function $v(x) = w(H^o(x))$, the anisotropic Hessian integral, recalling (3.5), is naturally defined as follows.

Definition 4.3. Let be v a concave function in $C^{0,1}(\mathcal{W}_R)$, such that v vanishes on $\partial\mathcal{W}_R$ and $v(x) = w(H^o(x)) = w(r)$. Then

$$I_H[v, \mathcal{W}_R] = \kappa \int_0^R |w'(t)|^3 dt. \quad (4.6)$$

In particular, for $v(x) = u^\star(x) = u^\star(2\kappa H^o(x))$, $x \in \Omega^\star$, $P_H(\Omega) = 2\kappa R$ performing a change of variable we have that for

$$I_H[u^\star, \Omega^\star] = 4\kappa^3 \int_0^{P_H(\Omega)} |(u^\star)'(s)|^3 ds. \quad (4.7)$$

The following Pólya-Szegö inequality for anisotropic Hessian integral holds:

Theorem 4.1. Let be $u \in \Phi_0(\Omega) \cap C^2(E_u)$. Then

$$I_H[u, \Omega] \geq I_H[u^\star, \Omega^\star]. \quad (4.8)$$

Moreover, if u is strictly concave, then the equality in (4.8) holds if and only if, up to a translation, $\Omega = \Omega^\star$ and $u = u^\star$.

Proof. Using (4.5) and the coarea formula, we get that, for $M = \max_{\bar{\Omega}} u$,

$$I_H[u, \Omega] = \frac{1}{2} \int_{\Omega} k_H(x) H^3(\nabla u) dx = \frac{1}{2} \int_0^M dt \int_{u=t} H^3(\nabla u) \frac{k_H(x)}{|\nabla u|} d\sigma. \quad (4.9)$$

Now observe that, by the Hölder inequality

$$\int_{u=t} H(\nabla u) \frac{k_H(x)}{|\nabla u|} d\sigma \leq \left(\int_{u=t} H^3(\nabla u) \frac{k_H(x)}{|\nabla u|} d\sigma \right)^{\frac{1}{3}} \left(\int_{u=t} \frac{k_H(x)}{|\nabla u|} d\sigma \right)^{\frac{2}{3}}, \quad (4.10)$$

then using the homogeneity of H and formulas (2.11) and (4.3), we have

$$\int_{u=t} H^3(\nabla u) \frac{k_H(x)}{|\nabla u|} d\sigma \geq \frac{\left(\int_{u=t} k_H(x) H(\nu) d\sigma \right)^3}{\left(\int_{u=t} \frac{k_H(x)}{|\nabla u|} d\sigma \right)^2} = \frac{8\kappa^3}{(-\lambda'_H(t))^2} = 8\kappa^3 [(-u^\star)'(\lambda_H(t))]^2.$$

Hence applying the above inequality in (4.9), performing the change of variable $s = \lambda_H(t)$ and recalling (4.7) we get (4.8).

Now suppose that u is strictly concave in Ω , and that equality in (4.8) holds. Then (4.10) becomes an equality, hence

$$H(\nabla u)|_{\{u=t\}} = c(t), \quad t \in [0, \max_{\bar{\Omega}} u[.$$

this implies, recalling (4.2), that

$$\lambda_H(t) = \int_{\{u=t\}} \frac{H(\nabla u)}{|\nabla u|} d\sigma = -c(t)\mu'(t), \quad t \in [0, \max_{\bar{\Omega}} u[.$$

and, by (4.3) and (2.11), that

$$\lambda'_H(t) = -\frac{2\kappa}{c(t)}, \quad t \in [0, \max_{\bar{\Omega}} u[.$$

Hence from the two equalities above we have

$$\lambda'_H(t)\lambda(t) = 2\kappa\mu'(t), \quad t \in [0, \max_{\bar{\Omega}} u[.$$

Integrating, and recalling that u is strictly concave in Ω , there is a unique point where the function u achieves its maximum, then we can integrate the above equality, obtaining that

$$\lambda_H(t)^2 = 4\kappa\mu(t).$$

Hence, equality occurs in the anisotropic isoperimetric inequality for all the level sets of u . Then, for any $t \in [0, \max_{\bar{\Omega}} u[$, the set $\{u > t\}$ is, up to a translation, homothetic to the Wulff shape. In particular, $\Omega = \Omega^\star$. Together with the fact that $H(\nabla u)$ is constant on $\{u = t\}$, it is possible to proceed as in [FV], obtaining that all the level sets have the same center and, up to a translation, $u = u^\star$. \square

Remark 4.2. We observe that if we do not assume that u is strictly concave, the equality sign can occur in the inequality (4.8) also if u is not radial and Ω is not a Wulff shape. For the sake of simplicity, we give an example in the Euclidean case, with $H(x) = |x|$. Let us consider a strictly convex, bounded open set Ω_0 with C^2 boundary, and let Ω be the set $\Omega_0 + \delta D$, where D is the unit disk of \mathbb{R}^2 centered at the origin, and $\delta > 0$. Let us consider the function

$$u(x) = \delta^3 - d(x)^3, \quad x \in \Omega,$$

where $d(x) = \text{dist}(x, \Omega_0) = \inf_{z \in \Omega_0} |x - z|$, with $x \in \mathbb{R}^2$ (see Figure 4.2). Then the convexity of Ω_0 implies that d is a convex function. Moreover, the smoothness of the boundary of Ω_0 guarantees that d is $C^2(\mathbb{R}^2 \setminus \Omega_0)$. Finally, $|\nabla d| = 1$ in $\mathbb{R}^2 \setminus \bar{\Omega}_0$ (for the properties of the distance function we refer the reader, for example, to [GT] and [Ro]). Hence $u \in \Phi_0(\Omega) \cap C^2(\Omega)$, $|\nabla u|$ is constant on every level set of u . Hence, being $k_H(x)$ positive on every level set of u , the inequality (4.10) becomes an equality. Then also in (4.8) the equality sign holds, even if u is not radially symmetric and Ω is not a ball.

5. COMPARISON RESULTS

In this section we use the symmetrization with respect to the anisotropic perimeter to prove comparison results between the solutions of suitable fully nonlinear elliptic equations. Let us consider the following problem involving the anisotropic Monge-Ampère operator:

$$\begin{cases} \det_H[u] = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where Ω is a convex, bounded, smooth open set of \mathbb{R}^2 and $f > 0$ belongs to $C^{0,\alpha}(\Omega)$.

We will consider strong solutions of problem (5.1), namely functions $u \in \Phi_0(\Omega)$ which satisfy

$$\det_H[u] = f(x) \text{ a.e. in } \Omega.$$

We stress that the positivity of f ensures that the function u has not flat zones, and, by the concavity of u , $\nabla u = 0$ only where u attains its maximum.

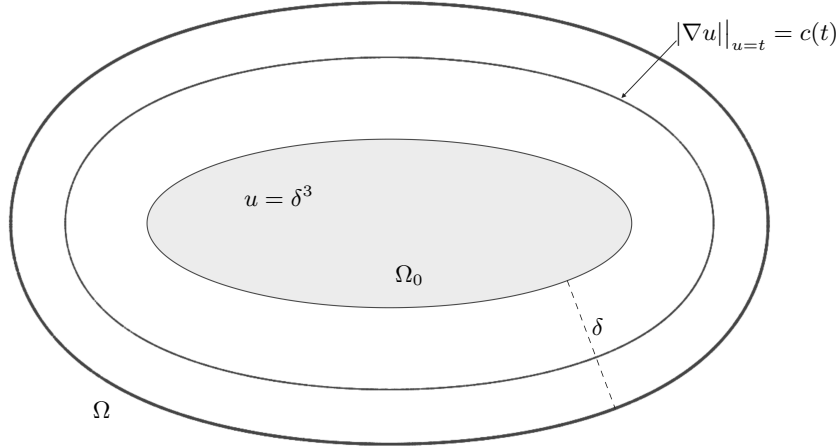


FIGURE 2. An example of the set $\Omega = \Omega_0 + \delta D$ of Remark 4.2. In Ω_0 the function u is constant, while on the curve $\{u = t\}$, $0 \leq t \leq \delta^3$, we have $|\nabla u| = c(t)$.

Remark 5.1. In the open set E_u , defined in (3.2), the operator $\det_H[\cdot]$ is continuous. Then a strong solution $u \in \Phi_0(\Omega)$ is a viscosity solution in E_u (see [Li, Corollary 3]). Moreover, if $x \in E_u$, then the equation in (5.1) can be written as

$$\det[\nabla^2 u(x)] = \frac{f(x)}{\det[\nabla_\xi^2 F(\nabla u(x))]}.$$

Hence, by the well-known regularity results for fully nonlinear elliptic equations (see [CC]), being $f \in C^{0,\alpha}(\Omega)$, and $f > 0$, then $u \in C^{2,\alpha}(E_u)$.

The following comparison result holds.

Theorem 5.1. *Let Ω be a convex, bounded, open set in \mathbb{R}^2 with C^2 boundary, and let $u \in \Phi_0(\Omega)$ be a strong solution of problem (5.1). Consider the unique anisotropic radially symmetric strong solution v of the symmetrized problem*

$$\begin{cases} \det_H[v] = f^\star(x) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star. \end{cases} \quad (5.2)$$

Then

$$u^\star \leq v \text{ in } \Omega^\star.$$

Proof. Let u be a strong solution of problem (5.1). As observed in Remark 5.1, $u \in C^2(E_u)$. Hence, its level sets Ω_t , $0 \leq t < \max_{\bar{\Omega}} u$ are C^2 and convex. Integrating both sides of the equation in (5.1), using (3.4), the Hölder inequality, (2.11) and (4.3) we get

$$\begin{aligned} \int_{u>t} f(x) dx &= \int_{u>t} \det_H[u] dx = \frac{1}{2} \int_{u=t} k_H(x) \frac{H(\nabla u)^3}{|\nabla u|} d\mathcal{H}^1 \geq \\ &\geq \frac{1}{2} \frac{\left(\int_{u=t} k_H(x) H(\nu) d\mathcal{H}^1 \right)^3}{\left(\int_{u=t} \frac{k_H(x)}{|\nabla u|} d\mathcal{H}^1 \right)^2} = \frac{4\kappa^3}{(-\lambda'_H(t))^2} = 4\kappa^3 [(-u^\star)'(\lambda_H(t))]^2, \end{aligned} \quad (5.3)$$

where last equality follows by the Definition 4.1 of symmetrization with respect to the anisotropic perimeter. By the well-known Hardy-Littlewood inequality and the anisotropic isoperimetric

inequality (2.8) we obtain

$$[(-u^*)'(\lambda_H(t))]^2 \leq \frac{1}{4\kappa^3} \int_0^{\mu(t)} f^*(r) dr \leq \frac{1}{4\kappa^3} \int_0^{\frac{\lambda_H^2(t)}{4\kappa}} f^*(r) dr.$$

Here we mean $f^*(s) = 0$ if $s \geq |\Omega|$. Performing the change of variable $s = \lambda_H(t)$ we get

$$[(-u^*)'(s)]^2 \leq \frac{1}{4\kappa^3} \int_0^{\frac{s^2}{4\kappa}} f^*(r) dr, \quad s \in [0, P_H(\Omega)]. \quad (5.4)$$

By (3.5) the unique anisotropic radially symmetric strong solution to (5.2), $v(x) = w(r)$, with $r = H^o(x)$ is

$$w(r) = \frac{1}{\sqrt{\kappa}} \int_r^R \left(\int_0^{\kappa r^2} f^*(t) dt \right)^{\frac{1}{2}} dr,$$

and then

$$v^*(s) = \frac{1}{2\kappa^{\frac{3}{2}}} \int_s^{P_H(\Omega)} \left(\int_0^{\frac{\sigma^2}{4\kappa}} f^*(t) dt \right)^{\frac{1}{2}} d\sigma \quad s \in [0, P_H(\Omega)]. \quad (5.5)$$

By (5.4) and (5.5) we get

$$u^*(s) \leq v^*(s), \quad s \in [0, P_H(\Omega)].$$

□

Acknowledgement. This work has been partially supported by the FIRB 2013 project “Geometrical and qualitative aspects of PDE’s”

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